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Comparison of Various Means for Operators

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For Hilbert space operators H, K, X with $H, K \geqslant 0$ the norm inequality $|||H^{1/2}XK^{1/2}||| \leqslant \frac{1}{2}\,|||HX+XK|||$ is known, where $|||\cdot|||$ is an arbitrary unitarily invariant norm. A refinement of this arithmetic–geometric mean inequality is studied. Similar norm inequalities are indeed established for various natural means for operators such as the logarithmic mean. © 1999 Academic Press

1. INTRODUCTION

The arithmetic–geometric mean inequality $\sqrt{ab} \leqslant \frac{1}{2}(a+b)(a, b \in \mathbf{R}_+)$ is of frequent use everywhere in mathematics. Its generalization to matrices and/or Hilbert space operators is tricky due to the lack of commutativity of involved operators. Indeed, even before proving something meaningful, a reasonable justification of "$\sqrt{HK}$" looks hopeless unless positive operators H, K commute. However, somewhat surprisingly, for operators H, K, X with $H, K \geqslant 0$ the (operator) norm inequality

$$\|H^{1/2}XK^{1/2}\| \leqslant \|\tfrac{1}{2}(HX+XK)\|$$

is valid as was first shown by A. McIntosh [13], and was used to study Heinz type inequalities [6] and some others. R. Bhatia and C. Davis [3] then succeeded in proving the same inequality for an arbitrary unitarily invariant norm $|||\cdot|||$. This arithmetic–geometric mean (norm) inequality has been recently under active investigation, and a very readable account

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(and quite a complete list of references) on this inequality as well as related topics can be found in [1, 2].

In the scalar case, besides the arithmetic and geometric means one has quite natural means such as

$$\frac{1}{n}\sum_{k=0}^{n-1} a^{k/(n-1)}b^{(n-1-k)/(n-1)}, \qquad \frac{1}{n}\sum_{k=1}^{n} a^{k/(n+1)}b^{(n+1-k)/(n+1)},$$

and the limit case (as $n \to \infty$) is the logarithmic mean

$$\int_0^1 a^t b^{1-t}\,dt = \frac{a-b}{\log a - \log b}.$$

Comparison among all of them is of course well-known and actually follows from a convexity argument. The purpose of the present article is to obtain its generalization in the operator setting. We prove the norm inequalities

$$|||H^{1/2}HK^{1/2}||| \leqslant \left|\left|\left| \int_0^1 H^t X K^{1-t}\,dt \right|\right|\right| \leqslant \tfrac{1}{2}\,|||HX+XK|||,$$

for example. Similar ones corresponding to the preceding two series of means are obtained, and furthermore comparison among these means and the logarithmic one is also made (Theorem 5). The operator integral defining the logarithmic mean can be understood in the weak sense, i.e., by considering the inner product. However, further discussion on this matter is presented in the Appendix.

In [11] the second-named author observed that the geometric mean can be written as a certain integral expression involving the arithmetic mean (see the beginning of the next section), from which the desired norm inequality follows immediately. It is actually the main idea behind the current article. On the other hand, the use of multipliers (see [9] for example) is one of the standard tools for studying matrix inequalities. In [4, 8, 12] this approach was indeed employed to investigate the arithmetic–geometric mean inequality among other things. This approach is closely related to ours, and in both of them establishing the positivity of relevant multiplier matrices or equivalently the positive definiteness of related functions is a crucial step.

Basic facts on the unitarily invariant norms and the corresponding symmetrically normed ideals (of compact operators) can be found in [5, 7, 14]. A more systematic and unified study on means for (finite) matrices will be published elsewhere.

2. GEOMETRIC, LOGARITHMIC, AND ARITHMETIC MEANS

Let $\mathscr{H}$ be a Hilbert space, and H, K, X be bounded linear operators on $\mathscr{H}$ throughout the article. We assume the positivity of H, K and set

$$G = H^{1/2} X K^{1/2} \qquad \text{(geometric mean)},$$

$$L = \int_0^1 H^t X K^{1-t}\, dt \qquad \text{(logarithmic mean)},$$

$$A = \tfrac{1}{2}(HX + XK) \qquad \text{(arithmetic mean)},$$

where the meaning of the operator integral for L will be clarified shortly. In [11] the integral expression

$$G = \int_{-\infty}^{\infty} H^{it} A K^{-it} \frac{dt}{2\cosh(\pi t)}$$

was pointed out. In this section we will obtain similar concrete integral expressions among the above three means.

When a (Lebesgue) integrable function $f(t)$ is given, we can define the operator

$$\int_{-\infty}^{\infty} H^{it} X K^{-it} f(t)\, dt$$

in the weak sense. (For a moment we assume the non-singularity of H, K for convenience.) In fact, the integral

$$\int_{-\infty}^{\infty} (H^{it} X K^{-it} \xi_1, \xi_2)\, f(t)\, dt$$

makes sense for each vectors $\xi_i \in \mathscr{H}$, and the numerical integral here is majorized by $\|X\|\, \|\xi_1\|\, \|\xi_2\| \times \int_{-\infty}^{\infty} |f(t)|\, dt$ in modulus. Therefore, the above operator integral determines a bounded operator (with norm less than $\|X\| \times \int_{-\infty}^{\infty} |f(t)|\, dt$) via the Riesz theorem. This is indeed the precise meaning of the above integral expression of G in terms of A. On the other hand, it is straightforward to observe that the map $t \in [0, \infty) \mapsto H^t$ is continuous in the strong operator topology, and hence the map $t \in [0, 1] \mapsto (H^t X K^{1-t} \xi_1, \xi_2) = (X K^{1-t} \xi_1, H^t \xi_2)$ is a continuous function for each $\xi \in \mathscr{H}$. Therefore, the logarithmic mean $L = \int_0^1 H^t X K^{1-t}$ makes perfect sense (in the weak sense); see also the discussions in the Appendix.

We begin with the case $H=K$, and in addition we assume that H is of the form

$$H=\sum_{i=1}^{l} \lambda_i P_i \tag{1}$$

with a mutually orthogonal family $\{P_i\}_{i=1,2,\ldots,l}$ of projections summing up to 1 and $0<\lambda_1<\lambda_2<\cdots<\lambda_l$. Observe

$$G=\sum_{i,j} g_{i,j} P_i X P_j, \qquad A=\sum_{i,j} a_{i,j} P_i X P_j$$

with

$$g_{i,j}=\lambda_i^{1/2}\lambda_j^{1/2}, \qquad a_{i,j}=\frac{\lambda_i+\lambda_j}{2}.$$

We also compute $L=\sum_{i,j} l_{i,j} P_i X P_j$ with

$$l_{i,j}=\int_0^1 \lambda_i^t \lambda_j^{1-t}\,dt=\lambda_j\times\frac{\frac{\lambda_i}{\lambda_j}-1}{\log\left(\frac{\lambda_i}{\lambda_j}\right)}=\frac{\lambda_i-\lambda_j}{\log\lambda_i-\log\lambda_j}.$$

Because the unitary H^{it} is $\sum_i \lambda_i^{it} P_i$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} H^{it} X H^{-it} g(t)\,dt &=\sum_{ij}\left(\int_{-\infty}^{\infty}\lambda_i^{it}\lambda_j^{-it} g(t)\,dt\right) P_i X P_j\\ &=\sum_{i,j}\left(\int_{-\infty}^{\infty} e^{it(\log\lambda_i-\log\lambda_j)} g(t)\,dt\right) P_i X P_j\\ &=\sum_{i,j}(\mathscr{F}g)(\log\lambda_i-\log\lambda_j)\, P_i X P_j\end{aligned}$$

with the Fourier transform $(\mathscr{F}g)(s)$. This (well-known) observation will be used repeatedly in what follows.

We now investigate how G and L are related. To do so, we look at the ratio

$$\begin{aligned}\frac{g_{i,j}}{l_{i,j}}&=\frac{\log\lambda_i-\log\lambda_j}{\lambda_i-\lambda_j}\times\lambda_i^{1/2}\lambda_j^{1/2}\\ &=\frac{\log\lambda_i-\log\lambda_j}{\lambda_i^{1/2}\lambda_j^{-1/2}-\lambda_i^{-1/2}\lambda_j^{1/2}}=\frac{\log\lambda_i-\log\lambda_j}{2\sinh\left(\frac{\log\lambda_i-\log\lambda_j}{2}\right)},\end{aligned}$$

which means

$$g_{i,j} = l_{i,j} \times f(\log \lambda_i - \log \lambda_j) \qquad \text{with} \qquad f(s) = \frac{s}{2\sinh\left(\frac{s}{2}\right)}. \tag{2}$$

By recalling the well-known formula

$$\mathscr{F}\left(\frac{\pi}{2\cosh^2(\pi t)}\right) = \frac{s}{2\sinh\left(\frac{s}{2}\right)}, \tag{3}$$

from (2) we conclude (as long as H is of the form (1))

$$H^{1/2}XH^{1/2} = \int_{-\infty}^{\infty} H^{it}\left(\int_0^1 H^s X H^{1-s}\, ds\right) H^{-it} \frac{\pi}{2\cosh^2(\pi t)}\, dt. \tag{4}$$

When H is a general non-singular positive operator, (by making use of the spectral decomposition and step functions) we can choose a sequence $\{H_n\}_{n=1,2,\ldots}$ of operators of the form (1) converging to H in norm. Then $\{H_n^{it}\}_{n=1,2,\ldots}$ converges to H^{it} strongly, and Lebesgue's dominated convergence theorem guarantees that the above integral expression (4) remains valid. For a general positive operator H, we consider the unitary H^{it} only on the support space of H (and zero on the orthogonal complement). Then, by cutting everything to the support space, (4) still remains valid. Finally, by making use of the famous 2×2 matrix trick, i.e., by comparing the $(1, 2)$-components of the expression (4) for the operators

$$\begin{pmatrix} H & 0 \\ 0 & K \end{pmatrix}, \qquad \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix},$$

we conclude

$$H^{1/2}XK^{1/2} = \int_{-\infty}^{\infty} H^{it}\left(\int_0^1 H^s X K^{1-s}\, ds\right) K^{-it} \frac{\pi}{2\cosh^2(\pi t)}\, dt. \tag{5}$$

We next investigate how L and A are related. We compute

$$\begin{aligned} \frac{l_{i,j}}{a_{i,j}} &= \frac{\lambda_i - \lambda_j}{\log \lambda_i - \log \lambda_j} \times \frac{2}{\lambda_i + \lambda_j} \\ &= \frac{2}{\log \lambda_i - \log \lambda_j} \times \frac{\left(\frac{\lambda_i}{\lambda_j}\right)^{1/2} - \left(\frac{\lambda_i}{\lambda_j}\right)^{-1/2}}{\left(\frac{\lambda_i}{\lambda_j}\right)^{1/2} + \left(\frac{\lambda_i}{\lambda_j}\right)^{-1/2}}, \end{aligned}$$

and hence we have

$$l_{i,j} = (2a_{i,j}) \times f(\log \lambda_i - \log \lambda_j) \qquad \text{with} \qquad f(s) = \frac{\tanh\left(\frac{s}{2}\right)}{s}.$$

Notice

$$\int_0^1 \frac{\sinh(\alpha s)}{\sinh(s)}\, d\alpha = \frac{\cosh(s) - 1}{s \sinh(s)} = \frac{2 \sinh^2\left(\frac{s}{2}\right)}{s \times 2 \sinh\left(\frac{s}{2}\right) \cos \mathrm{h}\left(\frac{s}{2}\right)} = \frac{\tanh\left(\frac{s}{2}\right)}{s}.$$

Since

$$\mathscr{F}\left(\frac{\sin(\pi\alpha)}{2(\cosh(\pi t) + \cosh(\pi\alpha))}\right) = \frac{\sinh(\alpha s)}{\sinh(s)} \qquad (0 < \alpha < 1), \tag{6}$$

we get

$$\begin{aligned}
&\int_0^1 H^s X H^{1-s}\, ds \\
&\quad = \int_0^1 \left(\int_{-\infty}^{\infty} H^{it}(HX + XH)\, H^{-it} \frac{\sin(\pi\alpha)}{2(\cosh(\pi t) + \cosh(\pi\alpha))}\, dt \right) d\alpha \\
&\quad = -\int_{-\infty}^{\infty} H^{it}(HX + XH)\, H^{-it} \log \left| \tanh\left(\frac{\pi t}{2}\right) \right| \frac{dt}{\pi}
\end{aligned}$$

(as long as H is of the form (1)) thanks to the Fubini theorem. In fact, by setting $x = \cos(\pi\alpha)$ we compute

$$\begin{aligned}
\int_0^1 \frac{\sin(\pi\alpha)}{2(\cosh(\pi t) + \cos(\pi\alpha))}\, d\alpha &= \frac{1}{2} \int_{-1}^1 \frac{1}{\cosh(\pi t) + x} \frac{d\alpha}{\pi} \\
&= \frac{1}{2\pi} \log \left(\frac{\cosh(\pi t) + 1}{\cosh(\pi t) - 1} \right) \\
&= \frac{1}{2\pi} \log \left(\frac{\cosh^2\left(\frac{\pi t}{2}\right)}{\sinh^2\left(\frac{\pi t}{2}\right)} \right) \\
&= -\frac{1}{\pi} \log \left| \tanh\left(\frac{\pi t}{2}\right) \right|.
\end{aligned}$$

Note that our computation means

$$\mathscr{F}\left(-\frac{1}{\pi}\log\left|\tanh\left(\frac{\pi t}{2}\right)\right|\right)=\frac{\tanh\left(\frac{s}{2}\right)}{s} \qquad \text{and}$$
$$-\int_{-\infty}^{\infty}\frac{1}{\pi}\log\left|\tanh\left(\frac{\pi t}{2}\right)\right|\,dt=\frac{1}{2}.$$

At any rate, by repeating the arguments before (5), we get

$$\int_0^1 H^s X K^{1-s}\,ds=-\int_{-\infty}^{\infty} H^{it}(HX+XK)\,K^{-it}\log\left|\tanh\left(\frac{\pi t}{2}\right)\right|\frac{dt}{\pi} \tag{7}$$

for general positive operators H, K.

We point out that (5) implies

$$|||H^{1/2}XK^{1/2}|||\leqslant\left|\left|\left|\int_0^1 H^s X K^{1-s}\,ds\right|\right|\right|$$

for any unitarily invariant norm $|||\cdot|||$. It is completely trivial if involved operators are finite matrices, but for Hilbert space operators more careful argument is needed. The required argument was actually presented in Appendix A of [11]. However, for the reader's convenience we repeat it in what follows. Let L be the logarithmic mean as usual, and the precise meaning of (5) is

$$\begin{aligned}(H^{1/2}XK^{1/2}\xi,\xi)&=\int_{-\infty}^{\infty}(H^{it}LK^{-it}\xi,\xi)\,\frac{\pi}{2\cosh^2(\pi t)}\,dt\\&=\lim_{n\to\infty}\int_{-n}^{n}(H^{it}LK^{-it}\xi,\xi)\,\frac{\pi}{2\cosh^2(\pi t)}\,dt\end{aligned}$$

for each vector $\xi\in\mathscr{H}$. This means that the operators

$$Y_n=\int_{-n}^{n} H^{it}LK^{-it}\,\frac{\pi}{2\cosh^2(\pi t)}\,dt \qquad (n=1,2,...)$$

tend to $H^{1/2}XK^{1/2}$ in the weak operator topology, and hence we get

$$|||H^{1/2}XK^{1/2}|||\leqslant\liminf_{n\to\infty}|||Y_n|||$$

thanks to the lower semi-continuity obtained in Proposition 2.11 of [7]. (When $|||\cdot|||=\|\cdot\|$, the operator norm, this lower semi-continuity follows simply from the expression $\|X\|=\sup\{|(X\xi_1,\xi_2)|:\|\xi_i\|\leqslant 1\}$.) Thus, (to get

$|||H^{1/2}XK^{1/2}||| \leqslant |||L|||$) it suffices to see $|||Y_n||| \leqslant |||L|||$ (for each n). For each fixed n, we set $\delta_m = 2n/m$ and consider the operator Riemann sum

$$Z_m = \delta_m \sum_{k=0}^{m-1} H^{i(-n+k\delta_m)} L K^{-i(-n+k\delta_m)} \frac{\pi}{2\cosh^2(\pi(-n+k\delta_m))}.$$

Since $\pi/2\cosh^2(\pi t)$ is continuous, so is the map $t \mapsto (H^{it}LK^{-it}\xi, \xi) \times \pi/2\cosh^2(\pi t)$ (for each $\xi \in \mathscr{H}$) and hence we have

$$\lim_{m\to\infty} (Z_m \xi, \xi) = (Y_n \xi, \xi).$$

This means the sequence $\{Z_m\}_{m=1,2,\ldots}$ tends to Y_n in the weak operator topology. Thus, the lower semi-continuity again implies

$$\begin{aligned}
|||Y_n||| &\leqslant \liminf_{m\to\infty} |||Z_m||| \\
&= \liminf_{m\to\infty} \left|\left|\left| \delta_m \sum_{k=0}^{m-1} H^{i(-n+k\delta_m)} L K^{-i(-n+k\delta_m)} \frac{\pi}{2\cosh^2(\pi(-n+k\delta_m))} \right|\right|\right| \\
&\leqslant |||L||| \times \liminf_{m\to\infty} \left(\delta_m \sum_{k=0}^{m-1} \frac{\pi}{2\cosh^2(\pi(-n+k\delta_m))} \right) \\
&= |||L||| \times \int_{-n}^{n} \frac{\pi\, dt}{2\cosh^2(\pi t)}.
\end{aligned}$$

Now $|||Y_n||| \leqslant |||L|||$ simply follows from

$$\int_{-n}^{n} \frac{\pi\, dt}{2\cosh^2(\pi t)} \leqslant \int_{-\infty}^{\infty} \frac{\pi\, dt}{2\cosh^2(\pi t)} = 1.$$

The integral expression (7) enables us to compare (the norms of) the geometric and logarithmic means. In fact, in this time we need to define "Y_n" by using the integral over $[-n, 1/n] \cup [1/n, n]$ because the density function $-1/\pi \log|\tanh(\pi t/2)|$ diverges at $t=0$. Then, since $-\int_{-\infty}^{\infty} \log |\tanh(\pi t/2)|\, dt/\pi = \frac{1}{2}$, an argument identical to the above shows

PROPOSITION 1. *Let H, K, X be Hilbert space operators with $H, K \geqslant 0$. For any unitarily invariant norm $|||\cdot|||$ we have*

$$|||H^{1/2}XK^{1/2}||| \leqslant \left|\left|\left| \int_0^1 H^s X K^{1-s}\, ds \right|\right|\right| \leqslant \tfrac{1}{2}\, |||HX + XK|||.$$

What was crucial in the above argument is the continuity of the density functions $\pi/2\cosh^2(\pi t)$ and $-1/\pi \log|\tanh(\pi t/2)|$ appearing in the integral expressions. In fact, the continuity made it possible to consider the "even partition" $\{-n+k\delta_m\}_{k=0,1,\dots,m}$ independent of the choice of a vector, which guaranteed the weak convergence of $\{Z_m\}_{m=1,2,\dots}$ to Y_n. The authors suspect that there may be some reasoning to justify the expected inequality free from the continuity of the relevant density functions. However, the continuity is readily available for those we will encounter afterward (although they are less explicit), and hence we need not be so serious about the problem at this point. In the next section we will repeatedly use the preceding argument based on the continuity.

3. MAIN RESULT

Here we deal with the two series of natural operator means corresponding to

$$\frac{1}{m}\sum_{k=0}^{m-1} a^{k/(m-1)}b^{(m-1-k)/(m-1)}, \qquad \frac{1}{n}\sum_{k=1}^{n} a^{k/(n+1)}b^{(n+1-k)/(n+1)}$$

mentioned in the Introduction. Note that $m=2$ and $n=1$ correspond to the arithmetic and geometric means, respectively. In this section a much more precise and general comparison result (than Proposition 1) will be obtained as the main theorem.

Let H, K, X be bounded operators with $H, K \geqslant 0$ as usual, and we first set

$$G(n)=\frac{1}{n}\sum_{k=1}^{n} H^{k/(n+1)}XH^{(n+1-k)/(n+1)} \qquad (n=1,2,\dots).$$

When $H=\sum_i \lambda_i P_i$ as in (1), we have $G(n)=\sum_{i,j} g(n)_{i,j}P_iXP_j$ with

$$\begin{aligned} g(n)_{i,j} &= \frac{1}{n}\sum_{k=1}^{n} \lambda_i^{k/(n+1)}\lambda_j^{(n+1-k)/(n+1)} \\ &= \frac{1}{n}\times\frac{\lambda_i^{1/(n+1)}\lambda_j^{n/(n+1)}\left(1-\left(\dfrac{\lambda_i}{\lambda_j}\right)^{n/(n+1)}\right)}{1-\left(\dfrac{\lambda_i}{\lambda_j}\right)^{1/(n+1)}} \end{aligned}$$

(which is equal to

$$\frac{1}{n}\times\frac{\lambda_j^{n/(n+1)}-\lambda_i^{n/(n+1)}}{\lambda_i^{-1/(n+1)}-\lambda_j^{-1/(n+1)}}).$$

Therefore, we compute the ratio between $g(n)_{i,\,j}$ and $g(m)_{i,\,j}$ as

$$\frac{g(n)_{i,\,j}}{g(m)_{i,\,j}}=\frac{m}{n}\times\frac{\lambda_i^{1/(n+1)}\lambda_j^{n/(n+1)}\left(1-\left(\frac{\lambda_i}{\lambda_j}\right)^{n/(n+1)}\right)}{1-\left(\frac{\lambda_i}{\lambda_j}\right)^{1/(n+1)}}$$

$$\times\frac{1-\left(\frac{\lambda_i}{\lambda_j}\right)^{1/(m+1)}}{\lambda_i^{1/(m+1)}\lambda_j^{m/(m+1)}\left(1-\left(\frac{\lambda_i}{\lambda_j}\right)^{m/(m+1)}\right)}$$

$$=\frac{m}{n}\times\frac{\lambda_i^{1/(n+1)-1/(m+1)}}{\lambda_j^{m/(m+1)-n/(n+1)}}\times\frac{\left(1-\left(\frac{\lambda_i}{\lambda_j}\right)^{n/(n+1)}\right)\left(1-\left(\frac{\lambda_i}{\lambda_j}\right)^{1/(m+1)}\right)}{\left(1-\left(\frac{\lambda_i}{\lambda_j}\right)^{1/(n+1)}\right)\left(1-\left(\frac{\lambda_i}{\lambda_j}\right)^{m/(m+1)}\right)}$$

$$=\frac{m}{n}\times\frac{\lambda_i^{1/(n+1)-1/(m+1)}}{\lambda_j^{m/(m+1)-n/(n+1)}}\times\frac{\left(\frac{\lambda_i}{\lambda_j}\right)^{(1/2)(n/(n+1)+1/(m+1))}}{\left(\frac{\lambda_i}{\lambda_j}\right)^{(1/2)(1/(n+1)+m/(m+1))}}$$

$$\times\frac{\left[\left(\left(\frac{\lambda_i}{\lambda_j}\right)^{-(1/2)(n/(n+1))}-\left(\frac{\lambda_i}{\lambda_j}\right)^{(1/2)(n/(n+1))}\right)\left(\left(\frac{\lambda_i}{\lambda_j}\right)^{-(1/2)(1/(m+1))}-\left(\frac{\lambda_i}{\lambda_j}\right)^{(1/2)(1/(m+1))}\right)\right]}{\left[\left(\left(\frac{\lambda_i}{\lambda_j}\right)^{-(1/2)(1/(n+1))}-\left(\frac{\lambda_i}{\lambda_j}\right)^{(1/2)(1/(n+1))}\right)\left(\left(\frac{\lambda_i}{\lambda_j}\right)^{-(1/2)(m/(m+1))}-\left(\frac{\lambda_i}{\lambda_j}\right)^{(1/2)(m/(m+1))}\right)\right]}.$$

We observe that the second and third factors in the last expression above cancel out because of

$$\frac{1}{2}\left(\frac{n}{n+1}+\frac{1}{m+1}\right)-\frac{1}{2}\left(\frac{1}{n+1}+\frac{m}{m+1}\right)$$

$$=\frac{n-m}{(n+1)(m+1)}$$

$$=\frac{1}{m+1}-\frac{1}{n+1}\left(=\frac{n}{n+1}-\frac{m}{m+1}\right).$$

Thus, the computations so far mean

$$\frac{g(n)_{i,j}}{g(m)_{i,j}}=\frac{m}{n}\times\varphi_{n,m}\left(\frac{\log\lambda_i-\log\lambda_j}{2}\right)$$

with

$$\varphi_{n,m}(s)=\frac{\sinh\left(\frac{1}{m+1}s\right)}{\sinh\left(\frac{1}{n+1}s\right)}\times\frac{\sinh\left(\frac{n}{n+1}s\right)}{\sinh\left(\frac{m}{m+1}s\right)}.$$

We now assume $m>n$. Then we have $1/(m+1)<1/(n+1)$, $n/(n+1)<m/(m+1)$, and hence $\varphi_{n,m}(s)$ is positive definite. The inverse Fourier transform $\psi_{n,m}(t)$ (see below) is a positive integrable function satisfying

$$\int_{-\infty}^{\infty}\psi_{n,m}(t)\,dt=\varphi_{n,m}(0)=\frac{\frac{1}{m+1}\times\frac{n}{n+1}}{\frac{1}{n+1}\times\frac{m}{m+1}}=\frac{n}{m},$$

and as before (i.e., the arguments before (5)) we get

Lemma 2. *When $m>n$, we have*

$$\frac{1}{n}\sum_{k=1}^{n}H^{k/(n+1)}XK^{(n+1-k)/(n+1)}$$
$$=\frac{m}{n}\int_{-\infty}^{\infty}H^{it/2}\left(\frac{1}{m}\sum_{k=1}^{m}H^{k/(m+1)}XK^{(m+1-k)/(m+1)}\right)K^{-it/2}\psi_{n,m}(t)\,dt.$$

The density function $\psi_{n,m}(t)$ in the lemma is actually the convolution product of the two positive functions

$$a(t)=(n+1)\times\frac{\sin\left(\pi\frac{n+1}{m+1}\right)}{2\left(\cosh(\pi(n+1)\,t)+\cos\left(\pi\frac{n+1}{m+1}\right)\right)},$$

$$b(t)=\frac{m+1}{m}\times\frac{\sin\left(\pi\frac{(m+1)\,n}{m(n+1)}\right)}{2\left(\cosh\left(\pi\frac{m+1}{m}t\right)+\cos\left(\pi\frac{(m+1)\,n}{m(n+1)}\right)\right)}$$

(thanks to (6)). Notice $\psi_{n,m}(t)=\int_{-\infty}^{\infty} a(x)\, b(t-x)\, dx$ and the integrand here is majorized by a constant multiple of $a(x)$ (due to the boundedness of $b(t)$). Hence, the continuity of $b(t)$ and Lebesgue's dominated convergence theorem guarantee the continuity of $\psi_{n,m}(t)$.

We set

$$G(\infty)=\int_0^1 H^s X H^{1-s}\, ds \qquad (=L, \text{ the logarithmic mean}),$$

and let us assume $H=\sum_i \lambda_i P_i$ as before. Then, as was seen in the previous section, we get $G(\infty)=\sum_{i,j} g(\infty)_{i,j} P_i X P_j$ with $g(\infty)_{i,j}=(\lambda_i-\lambda_j)/(\log \lambda_i - \log \lambda_j)$. We observe

$$\frac{g(n)_{i,j}}{g(\infty)_{i,j}}=\frac{1}{n}\times \varphi_{n,\infty}\left(\frac{\log \lambda_i-\log \lambda_j}{2}\right)$$

with

$$\varphi_{n,\infty}(s)=\frac{s}{\sinh\left(\dfrac{1}{n+1}s\right)}\times\frac{\sinh\left(\dfrac{n}{n+1}s\right)}{\sinh(s)}$$

in the present case. It can be done by a direct computation, but it is also seen by computing $\lim_{m\to\infty} g(n)_{i,j}/g(m)_{i,j}$. The above $\varphi_{n,\infty}(s)$ is positive definite, and the inverse Fourier transform $\psi(t)_{n,\infty}$ is a positive integrable function satisfying

$$\int_{-\infty}^{\infty} \psi_{n,\infty}(t)\, dt=\varphi_{n,\infty}(0)=\frac{\dfrac{n}{n+1}}{\dfrac{1}{n+1}}=n.$$

Actually (3) and (6) show

$$\psi_{n,\infty}=\left(\frac{(n+1)^2\,\pi}{4\cosh^2\left(\pi\dfrac{n+1}{2}t\right)}\right)*\left(\frac{\sin\left(\pi\dfrac{n}{n+1}\right)}{2\left(\cosh(\pi t)+\cos\left(\pi\dfrac{n}{n+1}\right)\right)}\right).$$

Therefore, the argument in the paragraph after Lemma 2 shows the continuity of $\psi_{n,\infty}(t)$, and we also get

LEMMA 3. *For each n we have*

$$\frac{1}{n}\sum_{k=1}^{n} H^{k/(n+1)}XK^{(n+1-k)/(n+1)}$$
$$=\frac{1}{n}\int_{-\infty}^{\infty} H^{it/2}\left(\int_0^1 H^sXK^{1-s}\,ds\right)K^{-it/2}\psi_{n,\infty}(t)\,dt.$$

We set

$$A(n)=\frac{1}{n}\sum_{k=0}^{n-1} H^{k/(n-1)}XH^{(n-1-k)/(n-1)} \qquad (n=2,3,...).$$

Let us assume $H=\sum_i \lambda_i P_i$ as before, and we get $A(n)=\sum_{i,j} a(n)_{i,j}P_iXP_j$ with

$$a(n)_{i,j}=\frac{1}{n}\sum_{k=0}^{n-1}\lambda_i^{k/(n-1)}\lambda_j^{(n-1-k)/(n-1)}=\frac{1}{n}\times\frac{\lambda_j\left(1-\left(\frac{\lambda_i}{\lambda_j}\right)^{n/(n-1)}\right)}{1-\left(\frac{\lambda_i}{\lambda_j}\right)^{1/(n-1)}}$$
$$=\frac{1}{n}\times\frac{\lambda_j^{n/(n-1)}\left(1-\left(\frac{\lambda_i}{\lambda_j}\right)^{n/(n-1)}\right)}{\lambda_j^{1/(n-1)}\left(1-\left(\frac{\lambda_i}{\lambda_j}\right)^{1/(n-1)}\right)}=\frac{1}{n}\times\frac{\lambda_i^{n/(n-1)}-\lambda_j^{n/(n-1)}}{\lambda_i^{1/(n-1)}-\lambda_j^{1/(n-1)}}.$$

Looking at the ratio, we observe

$$a(m)_{i,j}=\frac{n}{m}\times\frac{\lambda_i^{m/(m-1)}-\lambda_j^{m/(m-1)}}{\lambda_i^{1/(m-1)}-\lambda_j^{1/(m-1)}}$$
$$\times\frac{\lambda_i^{1/(n-1)}-\lambda_j^{1/(n-1)}}{\lambda_i^{n/(n-1)}-\lambda_j^{n/(n-1)}}\times a(n)_{i,j}.$$

Note

$$\frac{m}{m-1}+\frac{1}{n-1}=\frac{n}{n-1}+\frac{1}{m-1}\quad(=k).$$

Dividing the above numerator and denominator by $\lambda_i^{k/2}\lambda_j^{k/2}$, we get

$$a(m)_{i,j}=\frac{n}{m}\times\frac{\sinh\left(\frac{m}{m-1}\left(\frac{\log\lambda_i-\log\lambda_j}{2}\right)\right)}{\sinh\left(\frac{1}{m-1}\left(\frac{\log\lambda_i-\log\lambda_j}{2}\right)\right)}$$

$$\times\frac{\sinh\left(\frac{1}{n-1}\left(\frac{\log\lambda_i-\log\lambda_j}{2}\right)\right)}{\sinh\left(\frac{n}{n-1}\left(\frac{\log\lambda_i-\log\lambda_j}{2}\right)\right)}\times a(n)_{i,j}.$$

The positive definiteness of the product here of the two quotients of sinh's is unclear in this form. However, we set

$$\begin{aligned}\varphi(s)&=\frac{\sinh((\alpha+1)\,s)}{\sinh(\alpha s)}\times\frac{\sinh(\beta s)}{\sinh((\beta+1)\,s)}-1\\&=\frac{\sinh((\alpha+1)\,s)\sinh(\beta s)-\sinh(\alpha s)\sinh((\beta+1)\,s)}{\sinh(\alpha s)\sinh((\beta+1)\,s)}\end{aligned}$$

with $\alpha=1/(m-1)$ and $\beta=1/(n-1)$. By substituting

$$\begin{cases}\sinh((\alpha+1)\,s)=\sinh(\alpha s)\cosh(s)+\cosh(\alpha s)\sinh(s),\\\sinh((\beta+1)\,s)=\sinh(\beta s)\cosh(s)+\cosh(\beta s)\sinh(s),\end{cases}$$

to the above, we compute

$$\begin{aligned}\varphi(s)&=\frac{\{\sinh(\beta s)\cosh(\alpha s)-\cosh(\beta s)\sinh(\alpha s)\}\sinh(s)}{\sinh(\alpha s)\sinh((\beta+1)\,s)}\\&=\frac{\sinh((\beta-\alpha)\,s)\sinh(s)}{\sinh(\alpha s)\sinh((\beta+1)\,s)}.\end{aligned}$$

Let $\varphi_n(s)$ be the function $\varphi(s)$ with $\alpha=1/(m-1)$, $\beta=1/(n-1)$, and $m=n+1$. In this case we observe $\alpha<\beta\leqslant 2\alpha$, that is, $0<\beta-\alpha\leqslant\alpha$ ($\beta=2\alpha$ occurs only when $n=2$). Hence $\varphi_n(s)$ is positive definite, and the computations so far mean

$$a(n+1)_{i,j}=\frac{n}{n+1}\left(1+\varphi_n\left(\frac{\log\lambda_i-\log\lambda_j}{2}\right)\right)a(n)_{i,j}.$$

By substituting this expression to $A(n+1)=\sum_{i,j} a(n+1)_{i,j} P_iXP_j$, we compute

$$
\begin{aligned}
A(n+1) &= \frac{n}{n+1}\left(\sum_{i,j} a(n)_{i,j} P_iXP_j + \sum_{i,j} a(n)_{i,j}\, \varphi_n\left(\frac{\log\lambda_i - \log\lambda_j}{2}\right) P_iXP_j\right) \\
&= \frac{n}{n+1} A(n) + \frac{n}{n+1} \sum_{i,j} a(n)_{i,j}\, \varphi_n\left(\frac{\log\lambda_i - \log\lambda_j}{2}\right) P_iXP_j \\
&= \frac{n}{n+1} A(n) + \frac{n}{n+1} \int_{-\infty}^{\infty} H^{it/2} A(n)\, H^{-it/2} \psi_n(t)\, dt
\end{aligned}
$$

with the inverse Fourier transform $\psi_n(t)$ of $\varphi_n(s)$. Therefore, we have shown

LEMMA 4. *We have*

$$
\begin{aligned}
&\frac{1}{n+1} \sum_{k=0}^{n} H^{k/n} X K^{(n-k)/n} \\
&\quad = \frac{n}{n+1}\left(\frac{1}{n}\sum_{k=0}^{n-1} H^{k/(n-1)} X K^{(n-1-k)/(n-1)}\right) \\
&\qquad + \frac{n}{n+1}\int_{-\infty}^{\infty} H^{it/2}\left(\frac{1}{n}\sum_{k=0}^{n-1} H^{k/(n-1)} X K^{(n-1-k)/(n-1)}\right) K^{-it/2} \psi_n(t)\, dt.
\end{aligned}
$$

Remark. For the means $A(n) = 1/n \sum_{k=0}^{n-1} H^{k/(n-1)} X K^{(n-1-k)/(n-1)}$ there is some ambiguity for the interpretation of the two extreme terms. Namely the 0th and $(n-1)$st terms could be understandable in two ways:

$$
\text{(i)}\quad s_{(H)}XK, \qquad HXs_{(K)}, \qquad \text{(ii)}\quad XK, \quad HX
$$

(with of course the coefficient $1/n$) where $s_{(H)}$, $s_{(K)}$ are the support projections of H, K, respectively. In either interpretation Lemma 4 is valid.

We have been dealing with (not necessarily non-singular) operators by cutting everything to the support space so that the lemma obviously holds under the first interpretation. To show the lemma under the second interpretation, we may and do assume $H=K$ (thanks to the 2×2 matrix trick used repeatedly), and the matrix notations

$$
H = \begin{pmatrix} H_0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}
$$

are more convenient for us. Notice

$$
H^{it} = \begin{pmatrix} H_0^{it} & 0 \\ 0 & 0 \end{pmatrix}
$$

by definition. Also notice that no $(2, 2)$-component shows up from either side of the integral expression in Lemma 4 due to the presence of some power of H everywhere (at least from the left or from the right). On the other hand, as was mentioned earlier, $(1, 1)$-components of the both sides agree. Hence, it remains to compare only the off-diagonal components. The contribution in the left side comes from the two extreme terms of $A(n+1)$ while that in the right side comes only from the two extreme terms of the first factor (i.e., $n/(n+1)\,A(n)$). They obviously agree because of $n/(n+1) \times 1/n = 1/(n+1)$, and hence we are done.

We note that the density function $\psi_n(t)$ in Lemma 3 satisfies

$$\int_{-\infty}^{\infty} \psi_n(t)\,dt = \varphi_n(0) = \frac{\beta-\alpha}{\alpha(\beta+1)} = \frac{1}{n}$$

(by recalling $\alpha = 1/(m-1)$, $\beta = 1/(n-1)$, and $m = n+1$). This function admits a convolution representation almost similar to that of $\psi_{n,m}(t)$ appearing in Lemma 2, and hence it is continuous.

We are now ready to prove the main result in the article.

THEOREM 5. *Let H, K, X be Hilbert space operators with $H, K \geqslant 0$, and $|||\cdot|||$ be a unitarily invariant norm.*

(i) *For each $m(\geqslant 1)$ and $n(\geqslant 2)$, the following inequalities are valid:*

$$|||H^{1/2}XK^{1/2}||| \leqslant \frac{1}{m}\left|\left|\left|\sum_{k=1}^{m} H^{k/(m+1)}XK^{(m+1-k)/(m+1)}\right|\right|\right| \leqslant \left|\left|\left|\int_0^1 H^t X K^{1-t}\,dt\right|\right|\right|$$

$$\leqslant \frac{1}{n}\left|\left|\left|\sum_{k=0}^{n-1} H^{k/(n-1)}XK^{(n-1-k)/(n-1)}\right|\right|\right| \leqslant \frac{1}{2}\,|||HK + XK|||.$$

(ii) *The quantity*

$$\frac{1}{m}\left|\left|\left|\sum_{k=1}^{m} H^{k/(m+1)}XK^{(m+1-k)/(m+1)}\right|\right|\right|$$

is monotone increasing in m, and furthermore we have the monotone convergence

$$\lim_{m\to\infty} \frac{1}{m}\left|\left|\left|\sum_{k=1}^{m} H^{k/(m+1)}XK^{(m+1-k)/(m+1)}\right|\right|\right| = \left|\left|\left|\int_0^1 H^t X K^{1-t}\,dt\right|\right|\right|.$$

(iii) *The quantity*

$$\frac{1}{n} |||\sum_{k=0}^{n-1} H^{k/(n-1)} X K^{(n-1-k)/(n-1)}|||$$

is monotone decreasing in n.

Proof. Recall that the continuity of the density functions (i.e., $\psi_{n,m}(t)$, $\psi_{n,\infty}(t)$, and $\psi_n(t)$) in the lemmas obtained so far has been already checked. Therefore, the integral expressions in these lemmas together with the argument in the last part of the previous section imply all the norm inequalities (including the monotonicity) except the third one in (i).

Because the map $t \in [0,1] \mapsto (H^t X K^{1-t} \xi_1, \xi_2)$ is continuous (for each $\xi_i \in \mathscr{H}$), the sequence $\{1/n \sum_{k=0}^{n-1} H^{k/(n-1)} X K^{(n-1-k)/(n-1)}\}_{n=2,3,\ldots}$ converges to $\int_0^1 H^t X K^{1-t}\, dt$ in the weak operator topology (in fact in the operator norm, see Proposition 6 or Proposition 7 in the Appendix) so that the lower semi-continuity [7] and the (already known) monotone decreasingness (in n) guarantee

$$\begin{aligned}
\left|\left|\left| \int_0^1 H^t X K^{1-t}\, dt \right|\right|\right| &\leqslant \liminf_{n\to\infty} \frac{1}{n} \left|\left|\left| \sum_{k=0}^{n-1} H^{k/(n-1)} X K^{(n-1-k)/(n-1)} \right|\right|\right| \\
&= \inf_n \frac{1}{n} \left|\left|\left| \sum_{k=0}^{n-1} H^{k/(n-1)} X K^{(n-1-k)/(n-1)} \right|\right|\right| \\
&\leqslant \frac{1}{n'} \left|\left|\left| \sum_{k=0}^{n'-1} H^{k/(n'-1)} X K^{(n'-1-k)/(n'-1)} \right|\right|\right|
\end{aligned}$$

for each n'. Hence, we get the third inequality in (i). (A direct proof is also possible.)

The sequence $\{1/m \sum_{k=1}^{m} H^{k/(m+1)} X K^{(m+1-k)/(m+1)}\}_{m=1,2,\ldots}$ also converges to $\int_0^1 H^t X K^{1-t}\, dt$ so that the lower semi-continuity and (the norm inequality obtained from) Lemma 3 imply

$$\begin{aligned}
\left|\left|\left| \int_0^1 H^t X K^{1-t}\, dt \right|\right|\right| &\leqslant \liminf_{m\to\infty} \frac{1}{m} \left|\left|\left| \sum_{k=1}^{m} H^{k/(m+1)} X K^{(m+1-k)/(m+1)} \right|\right|\right| \\
&\leqslant \limsup_{m\to\infty} \frac{1}{m} \left|\left|\left| \sum_{k=1}^{m} H^{k/(m+1)} X K^{(m+1-k)/(m+1)} \right|\right|\right| \\
&\leqslant \left|\left|\left| \int_0^1 H^t X K^{1-t}\, dt \right|\right|\right|,
\end{aligned}$$

which is the second assertion of (ii). ∎

A few remarks are in order:

(i) The value "$+\infty$" is admissible in the inequalities in the theorem (as well as in Proposition 1). For instance, the norm inequality $|||L||| \leqslant |||A|||$ means: (a) If the arithmetic mean A belongs to the symmetrically normed ideal corresponding to $|||\cdot|||$ (i.e., $|||A||| < \infty$) then so does the logarithmic mean L, and (b) furthermore the norm inequality holds.

(ii) The integral expression in Lemma 4 actually shows the following somewhat better estimate:

$$\left|\left|\left|\frac{1}{n+1} \sum_{k=0}^{n} H^{k/n} X K^{(n-k)/n} - \frac{n}{n+1} \left(\frac{1}{n} \sum_{k=0}^{n-1} H^{k/(n-1)} X K^{(n-1-k)/(n-1)}\right)\right|\right|\right|$$
$$\leqslant \frac{1}{n+1} \left|\left|\left|\frac{1}{n} \sum_{k=0}^{n-1} H^{k/(n-1)} X K^{(n-1-k)/(n-1)}\right|\right|\right|.$$

APPENDIX: CONVERGENCE OF MEANS TO THE LOGARITHMIC MEAN

Let $L = \int_0^1 H^t X K^{1-t}\, dt$ be the logarithmic mean, and as before, we set

$$G(m) = \frac{1}{m} \sum_{k=1}^{m} H^{k/(m+1)} X K^{(m+1-k)/(m+1)},$$

$$A(n) = \frac{1}{n} \sum_{k=0}^{n-1} H^{k/(n-1)} X K^{(n-1-k)/(n-1)}.$$

We observed $\lim_{m \to \infty} |||G(m)||| = |||L|||$ in Theorem 5. In this appendix we will see that in many circumstances the sequences $\{G(m)\}_{m=1,2,\ldots}$ and $\{A(n)\}_{n=2,3,\ldots}$ actually tend to L in the symmetrically normed ideal in question.

In the first result the concept of the regularity (called "mononormalizing" in [5]) of a norm is used to guarantee approximation by finite-rank operators. This notion corresponds to the separability of the relevant symmetrically normed ideal (see [5, 14] for details). Typical examples of regular unitarily invariant norms are C_p-norms $(1 \leqslant p < \infty)$

$$\|X\|_p = \left(\sum_{i=1}^{\infty} \mu_i(X)^p\right)^{1/p} \quad (=(Tr\,|X|^p)^{1/p}),$$

where $\{\mu_i(\cdot)\}_{i=1,2,\ldots}$ denotes the singular numbers (see [5, 7, 14]).

PROPOSITION 6. *Assume that a unitarily invariant norm $|||\cdot|||$ is either regular or equivalent to the operator norm $\|\cdot\|$. If $|||H|||<+\infty$ and $|||K|||<+\infty$, then we have the convergence*

$$\lim_{m\to\infty}|||G(m)-L|||=0 \qquad \textit{and} \qquad \lim_{n\to\infty}|||A(n)-L|||=0.$$

Proof. We may and do assume $|||H|||\leqslant 1$, $|||K|||\leqslant 1$, and $\|X\|\leqslant 1$ to prove the first convergence (the proof of the second is similar). The Hölder-type inequality

$$|||H^tXK^{1-t}|||\leqslant |||HX|||^t\,|||XK|||^{1-t}\leqslant \|X\|\;|||H|||^t\,|||K|||^{1-t} \qquad (t\in(0,1)) \quad (8)$$

(see [10] or [2, IX.5]) will be repeatedly used in the proof.

Choose and fix $\delta\in(0,\frac{1}{2})$ and $\varepsilon\in(0,1)$. We divide the summation $\sum_{k=1}^m$ into three parts according to $k/(m+1)<\delta$, $\delta\leqslant k/(m+1)\leqslant 1-\delta$ or $k/(m+1)>1-\delta$ and do the same for the integral $\int_0^1$. Note $|||H^tXK^{1-t}|||\leqslant 1$ due to (8). Hence, by counting the "length" of each of the four outside terms (in $\sum_{k=1}^m$ and $\int_0^1$), we estimate

$$\left|\left|\left|\frac{1}{m}\sum_{k=1}^m H^{k/(m+1)}XK^{(m+1-k)/(m+1)}-\int_0^1 H^tXK^{1-t}\,dt\right|\right|\right|$$
$$\leqslant 6\delta+\left|\left|\left|\frac{1}{m}\sum_k{}' H^{k/(m+1)}XK^{(m+1-k)/(m+1)}-\int_\delta^{1-\delta} H^tXK^{1-t}\,dt\right|\right|\right|,$$

where $\sum'$ means the summation over integers k in $[(m+1)\delta,(m+1)(1-\delta)]$.

Let P_ε be the spectral projection of H corresponding to the interval $[0,\varepsilon]$, and we decompose $H=H_\varepsilon+\tilde{H}_\varepsilon$ with $H_\varepsilon=HP_\varepsilon$ and $\tilde{H}_\varepsilon=H(1-P_\varepsilon)$. We similarly decompose $K=K_\varepsilon+\tilde{K}_\varepsilon$ by using the spectral projection of K. Then, the two terms in the right side of the above inequality are

$$\sum_k{}' H^{k/(m+1)}XK^{(m+1-k)/(m+1)}$$
$$=\sum_k{}' H^{k/(m+1)}XK^{(m+1-k)/(m+1)}+\sum_k{}' \tilde{H}_\varepsilon^{k/(m+1)}XK_\varepsilon^{(m+1-k)/(m+1)}$$
$$+\sum_k{}' \tilde{H}_\varepsilon^{k/(m+1)}X\tilde{K}_\varepsilon^{(m+1-k)/(m+1)},$$

$$\int_\delta^{1-\delta} H^tXK^{1-t}\,dt$$
$$=\int_\delta^{1-\delta} H_\varepsilon^tHK^{1-t}\,dt+\int_\delta^{1-\delta}\tilde{H}_\varepsilon^tXK_\varepsilon^{1-t}\,dt+\int_\delta^{1-\delta}\tilde{H}_\varepsilon^tX\tilde{K}_\varepsilon^{1-t}\,dt.$$

To estimate the first two terms in the above right sides, we first notice

$$|||H_\varepsilon||| \leqslant |||H|||, \qquad |||\tilde{H}_\varepsilon||| \leqslant |||H|||, \qquad |||K_\varepsilon||| \leqslant |||K|||$$

because of $H_\varepsilon = HP_\varepsilon$, etc. Hence, by the assumption all of the above are majorized by 1. Since $\delta \leqslant t, 1-t$ for $t \in [\delta, 1-\delta]$, the Hölder-type inequality (8) shows

$$|||H_\varepsilon^t X K^{1-t}||| \leqslant |||H_\varepsilon|||^t \leqslant |||H_\varepsilon|||^\delta, \qquad |||\tilde{H}_\varepsilon^t X K_\varepsilon^{1-t}||| \leqslant |||K_\varepsilon|||^{1-t} \leqslant |||K_\varepsilon|||^\delta$$

(as long as $t \in [\delta, 1-\delta]$). Therefore, we estimate

$$\begin{aligned}
&\left|\left|\left| \frac{1}{m} \sum_k{}' H^{k/(m+1)} X K^{(m+1-k)/(m+1)} - \int_\delta^{1-\delta} H^t X K^{1-t}\, dt \right|\right|\right| \\
&\quad \leqslant 2\, |||H_\varepsilon|||^\delta + 2\, |||K_\varepsilon|||^\delta + \left|\left|\left| \frac{1}{m} \sum_k{}' \tilde{H}_\varepsilon^{k/(m+1)} X \tilde{K}_\varepsilon^{(m+1-k)/(m+1)} \right.\right.\right. \\
&\qquad \left.\left.\left. - \int_\delta^{1-\delta} \tilde{H}_\varepsilon^t X \tilde{K}_\varepsilon^{1-t}\, dt \right|\right|\right|.
\end{aligned}$$

We observe that the map $t \in [\delta, 1-\delta] \mapsto \tilde{H}_\varepsilon^t X \tilde{K}_\varepsilon^{1-t}$ is continuous in norm $|||\cdot|||$. This is obvious when $|||\cdot|||$ is equivalent to the operator norm. Otherwise, H, K are compact operators and $\tilde{H}_\varepsilon, \tilde{K}_\varepsilon$ live on finite-dimensional subspaces so that the assertion is once again obvious. The continuity enables us to regard the integral $\int_\delta^{1-\delta} \tilde{H}_\varepsilon^t X \tilde{K}_\varepsilon^{1-t}\, dt$ as a ("vector-valued") Riemann integral, and $1/m \sum_k{}' \tilde{H}^{k/(m+1)} X \tilde{K}^{(m+1-k)/(m+1)}$ is a Riemann sum. Therefore, we conclude

$$\lim_{m\to\infty} \left|\left|\left| \frac{1}{m} \sum_k{}' \tilde{H}_\varepsilon^{k/(m+1)} X \tilde{K}_\varepsilon^{(m+1-k)/(m+1)} - \int_\delta^{1-\delta} \tilde{H}_\varepsilon^t X \tilde{K}_\varepsilon^{1-t}\, dt \right|\right|\right| = 0.$$

The estimates we obtained so far imply

$$\begin{aligned}
&\limsup_{m\to\infty} \left|\left|\left| \frac{1}{m} \sum_{k=1}^m H^{k/(m+1)} X K^{(m+1-k)/(m+1)} - \int_0^1 H^t X K^{1-t}\, dt \right|\right|\right| \\
&\quad \leqslant 6\delta + 2\, |||H_\varepsilon|||^\delta + 2\, |||K_\varepsilon|||^\delta.
\end{aligned}$$

Since δ, ε are arbitrary, it remains to see $|||H_\varepsilon|||, |||K_\varepsilon||| \to 0$ as $\varepsilon \to 0$. However, this follows from Proposition 2.12 of [7] for regular norms while it is trivial for the norm equivalent to the operator norm. ∎

When $|||X||| < \infty$ is assumed, the regularity for a unitarily invariant norm is irrelevant.

Proposition 7. *If $|||X||| < \infty$ for a unitarily invariant norm $|||\cdot|||$, then we have the convergence*

$$\lim_{m\to\infty} |||G(m)-L|||=0 \qquad \textit{and} \qquad \lim_{n\to\infty} |||A(n)-L|||=0.$$

The main idea of the proof is almost identical to the preceding proposition, but the role of (8) is replaced by the more elementary inequality

$$|||H^t X K^{1-t}||| \leqslant \|H^t\|\, |||X|||\ \|K^{1-t}\| = \|H\|^t\, |||X|||\, \|K\|^{1-t}. \tag{9}$$

We will just sketch a proof and the full details are left to the reader. In what follows the notations (such as $\sum_k'$, H_ε, $\tilde{H}_\varepsilon$, etc.) in the proof of Proposition 6 will be used.

We assume $\|H\|\leqslant 1$, $\|K\|\leqslant 1$, $|||X|||\leqslant 1$ and show just the first convergence. Choose and fix $\delta\in(0,\frac{1}{2})$, $\varepsilon\in(0,1)$. We observe $|||H^tXK^{1-t}|||\leqslant 1$ thanks to (9) and get

$$\begin{aligned}&\left|\left|\left|\frac{1}{m}\sum_{k=1}^m H^{k/(m+1)}XK^{(m+1-k)/(m+1)}-\int_0^1 H^tXK^{1-t}\,dt\right|\right|\right|\\&\qquad\leqslant 6\delta+\left|\left|\left|\frac{1}{m}\sum_k{}' H^{k/(m+1)}XK^{(m+1-k)/(m+1)}-\int_\delta^{1-\delta} H^tXK^{1-t}\,dt\right|\right|\right|\end{aligned}$$

as before. Since $\|H_\varepsilon\|, \|K_\varepsilon\|\leqslant\varepsilon$, by making use of (9) again we also estimate

$$\begin{aligned}&\left|\left|\left|\frac{1}{m}\sum_k{}' H^{k/(m+1)}XK^{(m+1-k)/(m+1)}-\int_\delta^{1-\delta} H^tXK^{1-t}\,dt\right|\right|\right|\\&\qquad\leqslant 4\varepsilon^\delta+\left|\left|\left|\frac{1}{m}\sum_k{}' \tilde{H}_\varepsilon^{k/(m+1)}X\tilde{K}_\varepsilon^{(m+1-k)/(m+1)}-\int_\delta^{1-\delta} \tilde{H}_\varepsilon^tX\tilde{K}_\varepsilon^{1-t}\,dt\right|\right|\right|.\end{aligned}$$

Recall that the maps $t\in[\delta,1-\delta]\mapsto\tilde{H}_\varepsilon^t$, $\tilde{K}_\varepsilon^{1-t}$ are $\|\cdot\|$-continuous. Hence, we get the continuity of the map $t\in[\delta,1-\delta]\mapsto\tilde{H}_\varepsilon^tX\tilde{K}_\varepsilon^{1-t}$ in the norm $|||\cdot|||$ based on the trivial fact $|||AXB|||\leqslant\|A\|\ |||X|||\ \|B\|$. This means that the interpretation of $\int_\delta^{1-\delta}$ as a Riemann integral is once again available. Therefore, as before we conclude

$$\limsup_{m\to\infty}\left|\left|\left|\frac{1}{m}\sum_{k=1}^m H^{k/(m+1)}XK^{(m+1-k)/(m+1)}-\int_0^1 H^tXK^{1-t}\,dt\right|\right|\right|\leqslant 6\delta+4\varepsilon^\delta,$$

which does the job.

Let

$$C_p=\{Y\in B(\mathscr{H});\ \|Y\|_p<\infty\}$$

be the Schatten class ($p \in [1, \infty)$). Note that one requires neither $H, K \in C_p$ nor $X \in C_p$ in the following result:

PROPOSITION 8. *If L belongs to the Schatten class C_p $(1 \leqslant p < \infty)$, then so do the means $G(m)$ and furthermore the sequence $\{G(m)\}_{m=1,2,\ldots}$ converges to L in the C_p-norm $\|\cdot\|$.*

Proof. As pointed out right after Theorem 5, we know $G(m) \in C_p$. The Ky Fan inequality (see [5, (2.12), p. 30]) and the convexity of $t \in \mathbf{R}_+ \mapsto t^p$ imply

$$\mu_{2i-1}(G(m)-L)^p \leqslant (\mu_i(g(m)) + \mu_i(L))^p \leqslant 2^{p-1}(\mu_i(G(m))^p + \mu_i(L)^p)$$

for each $i \in \{1, 2, \ldots\}$. We set

$$x_{i,m} = 2^{p-1}(\mu_i(G(m))^p + \mu_i(L)^p) - \mu_{2i-1}(G(m)-L)^p \qquad (\geqslant 0).$$

Recall the decreasingness

$$\mu_{2i-1}(G(m)-L) \leqslant \mu_1(G(m)-L) = \|G(m)-L\| \qquad \text{and}$$
$$|\mu_i(G(m)) - \mu_i(L)| \leqslant \|G(m)-L\|$$

(see [5, Corollary 2.3, Chap. 2]). Since $G(m)$ tends to L in the operator norm $\|\cdot\|$ (Proposition 6 or Proposition 7), the above inequalities imply

$$\lim_{m\to\infty} \mu_i(G(m)) = \mu_i(L) \qquad \text{and} \qquad \lim_{m\to\infty} \mu_{2i-1}(G(m)-L) = 0,$$

and hence we have

$$\lim_{m\to\infty} x_{i,m} = 2^p \mu_i(L)^p \qquad \text{(for each } i\text{)}.$$

Therefore, we estimate

$$\begin{aligned}
2^p \|L\|_p^p &= \sum_{i=1}^{\infty} 2^p \mu_i(L)^p = \sum_{i=1}^{\infty} \lim_{m\to\infty} x_{i,m} \leqslant \liminf_{m\to\infty} \sum_{i=1}^{\infty} x_{i,m} \\
&= \liminf_{m\to\infty} \left(2^{p-1}(\|G(m)\|_p^p + \|L\|_p^p) - \sum_{i=1}^{\infty} \mu_{2i-1}(G(m)-L)^p \right) \\
&= 2^p \|L\|_p^p + \liminf_{m\to\infty} \left(-\sum_{i+1}^{\infty} \mu_{2i-1}(G(m)-L)^p \right) \\
&= 2^p \|L\|_p^p - \limsup_{m\to\infty} \sum_{i=1}^{\infty} \mu_{2i-1}(G(m)-L)^p.
\end{aligned}$$

Here, we have used the already known fact $\lim_{m\to\infty} \|G(m)\|_p = \|L\|_p$ (Theorem 5 (ii)). By subtracting $2^p \|L\|_p^p$ $(<\infty)$ from both sides, we conclude

$$\limsup_{m\to\infty} \sum_{i=1}^{\infty} \mu_{2i-1}(G(m)-L)^p \leqslant 0, \qquad \text{that is,}$$

$$\lim_{m\to\infty} \sum_{i=1}^{\infty} \mu_{2i-1}(G(m)-L)^p = 0.$$

Now the desired result follows from the following trivial inequality (which is a consequence of the decreasingness of singular numbers):

$$\begin{aligned}\sum_{i=1}^{\infty} \mu_i(G(m)-L)^p &= \sum_{i=1}^{\infty} \mu_{2i-1}(G(m)-L)^p + \sum_{i=1}^{\infty} \mu_{2i}(G(m)-L)^p \\ &\leqslant 2 \sum_{i=1}^{\infty} \mu_{2i-1}(G(m)-L)^p. \quad \blacksquare\end{aligned}$$

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